1. (10 points) Verify that the straight lines $\mathbf{r}(t) = (3-t, 1+2t, t-1)$ and $\mathbf{r}(s) = (-s-2, -2s-1, s+4)$ intersect. Find the point of intersection and the equation of the plane that passes through both the straight lines.

To find the intersection point, we solve

$(3-t, 1+2t, t-1) = (-s-2, -2s-1, s+4)$

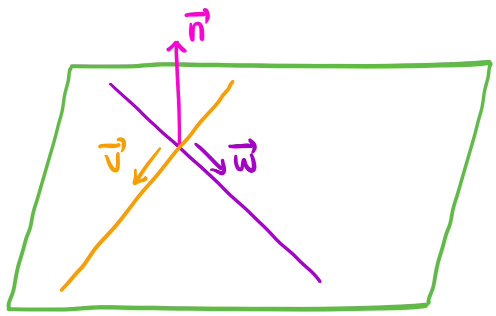
$\Rightarrow 3-t = -s-2, \; 1+2t = -2s-1, \; t-1 = s+4.$

1st equation : $3-t = -s-2 \Rightarrow s = t-5.$

2nd equation : $1+2t = -2(t-5)-1 \Rightarrow t=2, \; s=-3$

Check 3rd equation : $t-1 = 1$ and $s+4 = 1.$

The point of intersection is $\boxed{(1,5,1)}$



The two lines have direction vectors

$\vec{v} = (-1,2,1)$ and $\vec{w} = (-1,-2,1)$

A normal vector of the plane is

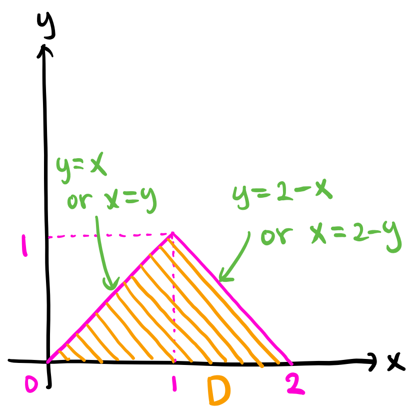
$\vec{n} = \vec{v} \times \vec{w} = (-1,2,1) \times (-1,-2,1) = (4,0,4)$

The plane is given by $\boxed{4(x-1) + 0\cdot(y-5) + 4(z-1) = 0}$

2. (10 points) Evaluate the double integral

$$\iint_D xy^2 \, dx \, dy,$$

where D is the triangular region with vertices at $(0,0)$, $(1,1)$, and $(2,0)$.



D is given by $0 \le y \le 1, \; y \le x \le 2-y.$

$$\iint_D xy^2 \, dA = \int_0^1 \int_y^{2-y} xy^2 \, dx \, dy = \int_0^1 \frac{x^2 y^2}{2} \Big|_{x=y}^{x=2-y} dy$$

$$= \frac{1}{2}\int_0^1 (2-y)^2 y^2 - y^4 \, dy = \frac{1}{2}\int_0^1 4y^2 - 4y^3 \, dy$$

$$= \frac{1}{2}\left(\frac{4}{3}y^3 - y^4\right)\Big|_{y=0}^{y=1} = \boxed{\frac{1}{6}}$$

3. Suppose $u(x,y)$ is a function of x, y and $x = r\cos\theta,\ y = r\sin\theta$.

(a) (3 points) Find $\frac{\partial u}{\partial r}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

$$\frac{\partial u}{\partial r} \underset{\uparrow}{=} \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \boxed{u_x\cos\theta + u_y\sin\theta}$$

chain rule

(b) (3 points) Find $\frac{\partial u}{\partial \theta}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

$$\frac{\partial u}{\partial \theta} \underset{\uparrow}{=} \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \boxed{-u_x r\sin\theta + u_y r\cos\theta}$$

chain rule

(c) (4 points) Find $\frac{\partial^2 u}{\partial \theta^2}$ in terms of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$, and $\frac{\partial^2 u}{\partial x \partial y}$.

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) \underset{\uparrow}{=} \frac{\partial}{\partial \theta}\left(-u_x r\sin\theta + u_y r\cos\theta\right)$$

(b)

$$\underset{\uparrow}{=} -\frac{\partial u_x}{\partial \theta}\cdot r\sin\theta - u_x r\cos\theta + \frac{\partial u_y}{\partial \theta}\cdot r\cos\theta - u_y r\sin\theta$$

product rule

$$\frac{\partial u_x}{\partial \theta} \underset{\uparrow}{=} \frac{\partial u_x}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u_x}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -u_{xx} r\sin\theta + u_{xy} r\cos\theta$$

chain rule

$$\frac{\partial u_y}{\partial \theta} \overset{\downarrow}{=} \frac{\partial u_y}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u_y}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -u_{xy} r\sin\theta + u_{yy} r\cos\theta$$

$$\Rightarrow \frac{\partial^2 u}{\partial \theta^2} = -\left(-u_{xx} r\sin\theta + u_{xy} r\cos\theta\right)\cdot r\sin\theta - u_x r\cos\theta$$

$$+\left(-u_{xy} r\sin\theta + u_{yy} r\cos\theta\right)\cdot r\cos\theta - u_y r\sin\theta$$

$$= \boxed{u_{xx} r^2\sin^2\theta - 2u_{xy} r^2\sin\theta\cos\theta + u_{yy} r^2\cos^2\theta - u_x r\cos\theta - u_y r\sin\theta}$$

<u>Note</u> You can give your answer in an unsimplified form on the exam.

4. (10 points) Find the maximum and minimum values of $f(x,y,z) = z$ on the surface $g(x,y,z) = x^2 + 2y^2 + 6z^2 - 2xz - 4yz - 12 = 0$.

Solve $\nabla f = \lambda \nabla g$ and $g = 0$

$\Rightarrow (0,0,1) = \lambda(2x-2z,\ 4y-4z,\ 12z-2x-4y)$

$\Rightarrow 0 = \lambda(2x-2z),\ 0 = \lambda(4y-4z),\ 1 = \underbrace{\lambda(12z-2x-4y)}_{\Rightarrow \lambda \neq 0}$

$\Rightarrow 0 = 2x-2z,\ 0 = 4y-4z,\ 1 = \lambda(12z-2x-4y)$

$\Rightarrow x = z,\ y = z.$

$g = 0 \Rightarrow x^2+2y^2+6z^2-2xz-4yz-12=0$

$\Rightarrow z^2+2z^2+6z^2-2z^2-4z^2-12=0 \Rightarrow 3z^2-12=0 \Rightarrow z = \pm 2.$

$\Rightarrow \boxed{\text{Maximum} = 2,\ \text{minimum} = -2}$

Note (1) Alternatively, you may write the surface equation as

$(x-z)^2 + 2(y-z)^2 + 3z^2 - 12 = 0$

$\Rightarrow 3z^2 = 12 - (x-z)^2 - 2(y-z)^2 \leq 12 \Rightarrow -2 \leq z \leq 2.$

(2) You can also solve this problem using implicit differentiation.

The surface equation $g=0$ defines z as an implicit function of x and y.

The extreme values of z occur at critical points.

$$\frac{\partial z}{\partial x} = -\frac{g_x}{g_z} = -\frac{2x-2z}{12z-2x-4y},\quad \frac{\partial z}{\partial y} = -\frac{g_y}{g_z} = -\frac{4y-4z}{12z-2x-4y}$$

$$\frac{\partial z}{\partial x} = 0,\ \frac{\partial z}{\partial y} = 0 \Rightarrow x = z,\ y = z.$$

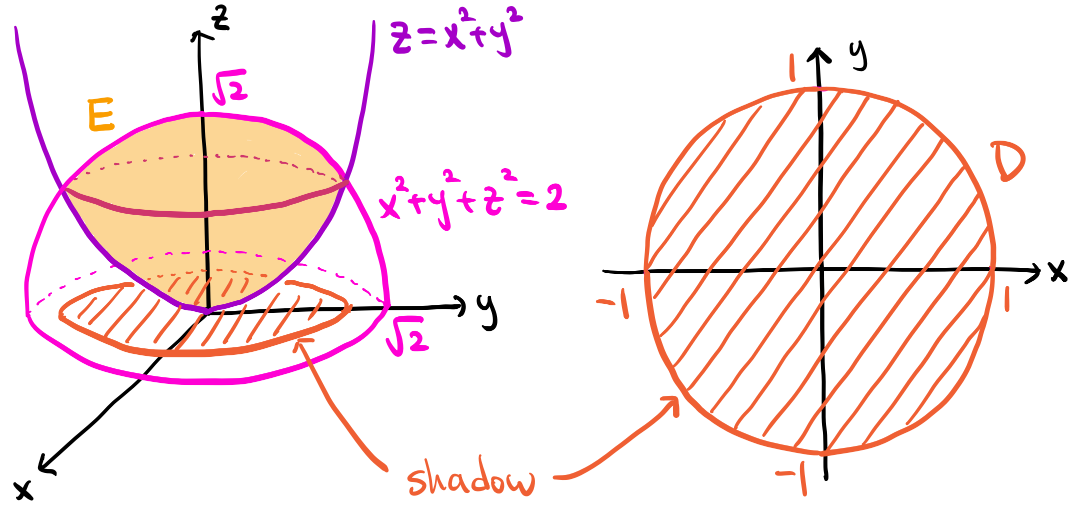
$g = 0 \Rightarrow x^2+2y^2+6z^2-2xz-4yz-12=0$

$\Rightarrow z^2+2z^2+6z^2-2z^2-4z^2-12=0 \Rightarrow 3z^2-12=0 \Rightarrow z = \pm 2$

$\Rightarrow$ Maximum $= 2$, minimum $= -2$

5. In this problem, you will find the volume of E, the bounded region between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$.

(a) (2 points) Sketch the region E approximately.



(b) (8 points) Find the volume of E.

In cylindrical coordinates :

$x^2+y^2+z^2 = 2 \rightsquigarrow r^2+z^2 = 2 \rightsquigarrow z = \sqrt{2-r^2}$ $(z \geq 0)$

$z = x^2+y^2 \rightsquigarrow z = r^2$

Intersection : $z = \sqrt{2-r^2}$ and $z = r^2 \Rightarrow z = \sqrt{2-z}$

$\Rightarrow z^2 = 2-z \Rightarrow z = 1, \cancel{-2} \Rightarrow r = 1$.

$z \geq 0$

The shadow on the xy-plane: $0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1$

For each point on the shadow : $r^2 \leq z \leq \sqrt{2-r^2}$.

$\Rightarrow 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1,\ r^2 \leq z \leq \sqrt{2-r^2}$

Jacobian

$$\text{Volume} = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} 1\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (\sqrt{2-r^2}-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_2^1 (u^{1/2}-2+u)(-\tfrac{1}{2})\,du\,d\theta = \int_0^{2\pi} -\frac{u^{3/2}}{3}+u-\frac{u^2}{4}\Bigg|_{u=2}^{u=1} d\theta$$

$u = 2-r^2$

$$= \int_0^{2\pi} \frac{8\sqrt{2}-7}{12}\,d\theta = \boxed{\frac{\pi(8\sqrt{2}-7)}{6}}$$

6. Consider the spherical shell $x^2 + y^2 + z^2 = 1$ with density $\rho(x, y, z) = z + 1$ (mass per unit area).

(a) (5 points) Find the total mass of the spherical shell.

Let S be the sphere $x^2+y^2+z^2=1$.

$m = \iint_S \rho(x,y,z)\,dS = \iint_S z+1\,dS = \iint_S z\,dS + \iint_S 1\,dS$

S is symmetric about the xy-plane $\Rightarrow \iint_S \underbrace{z}_{\text{odd w.r.t } z}\,dS = 0$

$\Rightarrow m = \iint_S 1\,dS = \text{Area}(S) = \underbrace{4\pi \cdot 1^2}_{\text{Area of sphere}} = \boxed{4\pi}$

(b) (5 points) Find the z coordinate of the center of mass.

$\bar{z} = \frac{1}{m}\iint_S z\rho(x,y,z)\,dS = \frac{1}{4\pi}\iint_S z(z+1)\,dS$

The unit normal vector of S is $\vec{n} = (x,y,z)$

(see Fact 2 in the Final exam facts note)

Let E be the solid ball $x^2+y^2+z^2 \le 1$.

The boundary $\partial E = S$ is oriented outward.

Set $\vec{F}(x,y,z) = (0,0,z+1) \Rightarrow \vec{F}\cdot\vec{n} = z(z+1)$ and $\text{div}(\vec{F}) = 1$.

$\iint_S z(z+1)\,dS = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S \vec{F}\cdot d\vec{S} = \iint_{\partial E} \vec{F}\cdot d\vec{S}$

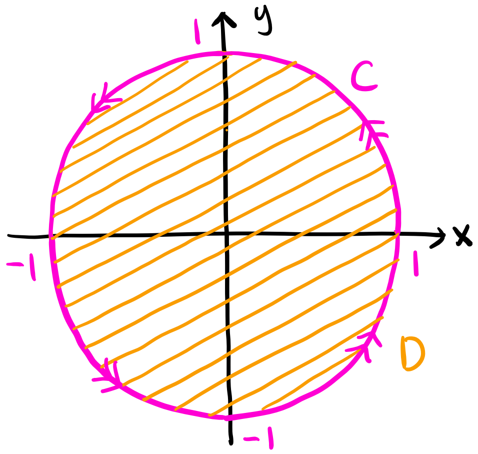
$\underset{\text{div.thm}}{=} \iiint_E \text{div}(\vec{F})\,dV = \text{Vol}(E) = \underbrace{\frac{4\pi}{3}\cdot 1^3}_{\text{Volume of sphere}} = \frac{4\pi}{3}$

$\Rightarrow \bar{z} = \frac{1}{4\pi}\iint_S z(z+1)\,dS = \frac{1}{4\pi}\cdot\frac{4\pi}{3} = \boxed{\frac{1}{3}}$

<u>Note</u> You can use the spherical parametrization

$\vec{r}(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ with $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$.

7. (10 points) Evaluate the line integral $\int_C y^3\,dx + (3x - x^3)\,dy$, where C is the curve $x^2 + y^2 = 1$ (counterclockwise sense), using Green's theorem.



D : the region bounded by C

$\Rightarrow \partial D = C$ is positively oriented.

$P = y^3,\ Q = 3x - x^3$

$$\Rightarrow \frac{\partial P}{\partial y} = 3y^2,\quad \frac{\partial Q}{\partial x} = 3 - 3x^2$$

$$\int_C y^3\,dx + (3x - x^3)\,dy = \int_{\partial D} P\,dx + Q\,dy \underset{\text{Green's thm}}{=} \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$= \iint_D 3 - 3x^2 - 3y^2\,dA$$

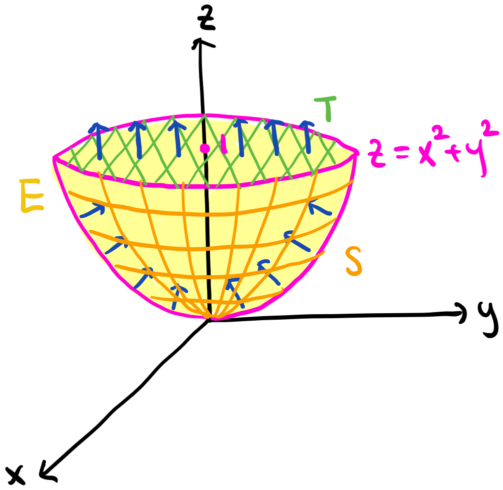
In polar coordinates, D is given by $0 \le \theta \le 2\pi,\ 0 \le r \le 1$

$$\Rightarrow \int_C y^3\,dx + (3x - x^3)\,dy = \int_0^{2\pi}\int_0^1 (3 - 3r^2)\underbrace{r}_{\text{Jacobian}}\,dr\,d\theta = \int_0^{2\pi} \left.\frac{3r^2}{2} - \frac{3r^4}{4}\right|_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \frac{3}{4}\,d\theta = \boxed{\frac{3\pi}{2}}$$

<u>Note</u> Even if the problem didn't tell you to use Green's theorem, you should still be able to recognize that this problem is about Green's theorem. After all, it is asking to evaluate a line integral over a loop for a vector field which is defined everywhere. Moreover, the given integral is very difficult to evaluate using a parametrization.

8. (a) (6 points) Find the flux of the vector field $\mathbf{F} = -x\mathbf{i} + z\mathbf{k}$ through the paraboloid surface $z = x^2 + y^2$, $0 \leq x^2 + y^2 \leq 1$. The surface is oriented upward.



S : the paraboloid $z = x^2+y^2$ with $x^2+y^2 \leq 1$, oriented upward.

T : the disk $x^2+y^2 \leq 1$ with $z=1$, oriented upward

E : the solid bounded by S and T.

$\Rightarrow \partial E = -S + T$ is oriented outward.

(S is oriented inward with respect to E)

$\vec{F} = (-x, 0, z) \Rightarrow \operatorname{div}(\vec{F}) = 0$.

$$\iint_{\partial E} \vec{F} \cdot d\vec{S} = -\iint_S \vec{F} \cdot d\vec{S} + \iint_T \vec{F} \cdot d\vec{S} \quad (*)$$

$$\iint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E \underbrace{\operatorname{div}(\vec{F})}_{=0} dV = 0$$

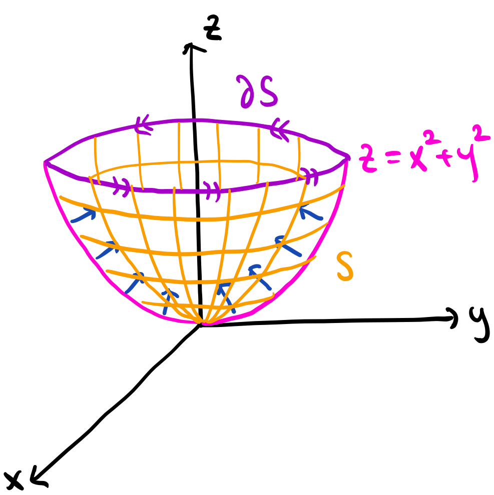
The unit normal vector of T is $\vec{n} = (0,0,1) \Rightarrow \vec{F} \cdot \vec{n} = z = 1$ on T.

$$\iint_T \vec{F} \cdot d\vec{S} = \iint_T \vec{F} \cdot \vec{n}\, dS = \iint_T 1\, dS = \text{Area}(T) = \pi \cdot 1^2 = \pi.$$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} \underset{(*)}{=} \iint_{\partial E} \vec{F} \cdot d\vec{S} + \iint_T \vec{F} \cdot d\vec{S} = \boxed{\pi}$$

<u>Note</u> You can also compute this integral using the xy-parametrization $\vec{r}(x,y) = (x, y, x^2+y^2)$ with $x^2+y^2 \leq 1$.

(b) (4 points) Let $\mathbf{v} = xz\mathbf{j}$. Find curl$\mathbf{v}$ and use Stokes's theorem to evaluate the line integral $\int_C \mathbf{v}.d\mathbf{r}$, where C is the boundary of the paraboloid surface above (with counterclockwise sense in the $z = 1$ plane).



$\partial S = C$ is positively oriented.

$\vec{v} = (0, xz, 0) \Rightarrow \operatorname{curl}(\vec{v}) = (-x, 0, z) = \vec{F}$.

$$\int_C \vec{v} \cdot d\vec{r} = \int_{\partial S} \vec{v} \cdot d\vec{r} \underset{\text{Stokes' thm}}{=} \iint_S \operatorname{curl}(\vec{v}) \cdot d\vec{S}$$

$$= \iint_S \vec{F} \cdot d\vec{S} \underset{(a)}{=} \boxed{\pi}$$

9. (10 points) Find the flux of the vector field $\mathbf{F}$ out of the sphere $x^2 + y^2 + z^2 = a^2$ in each case below. You can either evaluate the flux directly or use the divergence theorem. The position vector is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(a) $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^2}$.

Let S be the sphere $x^2+y^2+z^2=a^2$.

$\Rightarrow$ The unit normal vector of S is $\vec{n} = \left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right)$

(see Fact 2 in the Final exam facts note)

$\vec{r} = (x, y, z)$, $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

$$\Rightarrow \vec{F} = \left(\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}\right)$$

$$\vec{F}\cdot\vec{n} = \frac{x^2}{a(x^2+y^2+z^2)} + \frac{y^2}{a(x^2+y^2+z^2)} + \frac{z^2}{a(x^2+y^2+z^2)} = \frac{x^2+y^2+z^2}{a(x^2+y^2+z^2)} = \frac{1}{a} \text{ on } S$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S \frac{1}{a} = \frac{1}{a}\text{Area}(S) = \frac{1}{a}\cdot \underbrace{4\pi a^2}_{\text{Area of sphere}} = \boxed{4\pi a}$$

<u>Note</u> You can also use the spherical parametrization

$\vec{r}(\theta, \varphi) = (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$ with $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$.

However, you cannot use the divergence theorem with the solid ball $x^2+y^2+z^2 \le 1$ as $\vec{F}$ is not defined at the origin.

(b) $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3}$.

$$\vec{F} = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$\vec{F}\cdot\vec{n} = \frac{x^2}{a(x^2+y^2+z^2)^{3/2}} + \frac{y^2}{a(x^2+y^2+z^2)^{3/2}} + \frac{z^2}{a(x^2+y^2+z^2)^{3/2}} = \frac{x^2+y^2+z^2}{a(x^2+y^2+z^2)^{3/2}}$$

$$\underset{\uparrow}{=} \frac{a^2}{a\cdot a^3} = \frac{1}{a^2} \text{ on } S$$

$x^2+y^2+z^2=a^2$ on S

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S \frac{1}{a^2} = \frac{1}{a^2}\text{Area}(S) = \frac{1}{a^2}\cdot \underbrace{4\pi a^2}_{\text{Area of sphere}} = \boxed{4\pi}$$

(c) $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^4}$.

$$\vec{F} = \left(\frac{x}{(x^2+y^2+z^2)^2}, \frac{y}{(x^2+y^2+z^2)^2}, \frac{z}{(x^2+y^2+z^2)^2} \right)$$

$$\vec{F}\cdot\vec{n} = \frac{x^2}{a(x^2+y^2+z^2)^2} + \frac{y^2}{a(x^2+y^2+z^2)^2} + \frac{z^2}{a(x^2+y^2+z^2)^2} = \frac{x^2+y^2+z^2}{a(x^2+y^2+z^2)^2}$$

$$\underset{\substack{\uparrow \\ x^2+y^2+z^2=a^2 \text{ on } S}}{=} \frac{a^2}{a\cdot a^4} = \frac{1}{a^3} \text{ on } S$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S \frac{1}{a^3} = \frac{1}{a^3}\text{Area}(S) = \frac{1}{a^3}\cdot \underbrace{4\pi a^2}_{\text{Area of sphere}} = \boxed{\frac{4\pi}{a}}$$

(d) $\mathbf{F} = 4x\mathbf{i} - y\mathbf{j} - 2z\mathbf{k}$

Let E be the solid ball $x^2+y^2+z^2 \le a^2$.

$\Rightarrow \partial E = S$ is oriented outward.

$\vec{F} = (4x, -y, -2z) \Rightarrow \operatorname{div}(\vec{F}) = 1$.

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_{\partial E} \vec{F}\cdot d\vec{S} \underset{\substack{\uparrow \\ \text{div. thm}}}{=} \iiint_E \operatorname{div}(\vec{F})\, dV = \text{Vol}(E) = \boxed{\frac{4\pi a^3}{3}}$$

(e) $\mathbf{F} = (xy + xz)\mathbf{i} - 2(xy + yz)\mathbf{j} + (2xz - yz + z^2/2)\mathbf{k}$

$\vec{F} = (xy + xz, -2xy - 2yz, 2xz - yz + \frac{z^2}{2}) \Rightarrow \operatorname{div}(\vec{F}) = 0$.

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_{\partial E} \vec{F}\cdot d\vec{S} \underset{\substack{\uparrow \\ \text{div. thm}}}{=} \iiint_E \underbrace{\operatorname{div}(\vec{F})}_{=0}\, dV = \boxed{0}$$